Int. J. Heat Mass Transfer. Vol. 19, pp. 757–764. Pergamon Press 1976. Printed in Great Britain

THE VARIATION OF TURBULENT PRANDTL AND SCHMIDT NUMBERS IN WAKES AND JETS

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(*Received* 14 *May* 1975)

Abstract—Consideration is given to experimentally determined values of turbulent and effective Prandtl and Schmidt numbers within free-turbulent flows. Particular attention is given to the variations in these moduli across flows of nearly uniform density. In highly intermittent regions the moduli may differ by a factor of two from the values pertaining in the body of the shear layer. Simple analytical models are used to investigate the role of intermittency and the dependence on molecular diffusivities and Reynolds number.

NOMENCLATURE

- B, value of eddy–diffusivity ratio in the turbulent body of the flow;
- c, fluctuation in concentration;
- C, mean value of concentration;
- C_1, C_2, empirical constants;
- D, molecular diffusivity of matter;
- E, value of eddy–diffusivity ratio in edge region (for $\gamma = \frac{1}{2}$);
- l_0, scale of width of free-turbulent flow;
- N, effective value of eddy–diffusivity ratio in outer, non-vortical fluid;
- Pr, molecular Prandtl number of fluid (ν/κ);
- Pr_e, effective Prandtl number;
- Pr_t, turbulent Prandtl number ($\varepsilon_m/\varepsilon_h$);
- Re, Reynolds number;
- Re_d, Reynolds number for pipe flow;
- Re_0, Reynolds number characteristic of free-turbulent flow ($U_0 l_0/\nu$);
- R_0, empirical constant characteristic of a species of free-turbulent flow;
- Sc, molecular Schmidt number of a contaminant in a fluid (ν/D);
- Sc_e, effective Schmidt number;
- Sc_t, turbulent Schmidt number ($\varepsilon_m/\varepsilon_D$);
- T, mean value of temperature;
- u, velocity fluctuation in axial direction;
- U, mean axial velocity;
- U_0, change in mean velocity across a shear layer;
- v, transverse velocity fluctuation;
- ε_D, eddy diffusivity of matter;
- ε_h, eddy diffusivity of enthalpy;
- ε_m, eddy diffusivity of momentum, eddy viscosity;
- γ, intermittency factor, the probability that the fluid at a particular point be turbulent or highly vortical;
- κ, molecular diffusivity of enthalpy;
- ν, molecular diffusivity of momentum, kinematic viscosity;
- θ, temperature fluctuation;
- $\Delta\phi$, $= |\phi - \phi_1|$, where ϕ is the time-averaged value of any property at a point in the flow, and ϕ_1 is the value approached in the outer flow at the section considered.

1. INTRODUCTION

THE MOST common way of relating variations of time-averaged velocity, temperature and contaminant concentration across a free-turbulent shear layer is through the introduction of uniform values for the turbulent Prandtl and Schmidt numbers. (These are the ratios of the eddy viscosity to the eddy diffusivities of enthalpy and the contaminant.) For round jets Pr_t, $Sc_t = 0.7$ is widely used; for other shear flows (mixing layers, plane jets, and wakes) values near Pr_t, $Sc_t = 0.5$ are usually adopted. The numbers quoted are appropriate to flows of essentially constant density, with the molecular Pr or Sc not too different from unity [1]. For more varied flows in which the density varies considerably, as a result of high velocities, large temperature differences or large variations in species concentration, wider ranges of the moduli Pr_t and Sc_t are encountered [2]. Predictions are almost invariably based on the assumptions that the moduli are not merely uniform throughout the flow, but are nearly independent of the molecular diffusivities and Reynolds number, so long as the flow remains turbulent.

The particular purpose of this paper is to inquire into the first of these assumptions. The roles of the molecular transport properties and Reynolds number are also of interest, but the available experimental data for free turbulence hardly justify definite conclusions. Nevertheless, the somewhat contradictory evidence does suggest that changes in Pr_t and Sc_t from flow to flow, consequent upon changes in Reynolds number and the molecular values Pr and Sc, are rather smaller than the changes across an individual flow. The reasons why this should be so will be considered in Section 4 below.

The most detailed of the measurements from which Pr_t and Sc_t can be determined are those relating to the flows most conveniently studied in the laboratory, namely, low-velocity jets and wakes in air, the shear flow either being moderately heated or containing a dilute gaseous contaminant. Fage and Falkner [3] and Reichardt [4] have studied the plane wake; Hall and Hislop [5] and Reichardt and Emshaus [6] have examined round wakes; van der Hegge Zijnen [7] and Davies *et al.* [8] have considered the plane jet; Pabst

[9], Hinze and van der Hegge Zijnen [10], and Corrsin and Uberoi [11] have made measurements in round jets. We shall depend for the most part on this body of data relating to essentially passive contaminants, with Pr or Sc not far from unity, in flows that develop in a nearly self-preserving manner. The final restriction is important not merely in providing a coherent body of measurements, but in providing the justification for the calculation of the ratios Pr_t and Sc_t from profiles of mean values, as is shown in the Appendix to this paper. Although only a few comparisons wll be made with measurements lying outside the class specified above, it is instructive to note some allied lines of investigation.

Our knowledge of transfers across turbulent mixing layers is more limited than that for jets and wakes. Watt [12] has studied velocity and temperature variations and fluctuations in a mixing layer in air, but the accuracy obtained was not sufficient for the confident prediction of turbulent Prandtl numbers. (For this species of flow, as is shown in the Appendix, it is not possible to determine Pr_t from the mean-value profiles alone; measurements of the double-velocity (shear-stress) correlation $\overline{uv}$ and the velocity–temperature (heat-flux) correlation $\overline{v\theta}$ are also required.) Rebollo [13] has studied velocity and concentration variations and fluctuations in mixing layers between streams of helium and nitrogen, and has determined variations of Sc_t from these data. However, this is a case in which large changes in density are found; indeed, this flow differs from pure gaseous jets in that continued mixing does not carry it towards the low-concentration limit.

Studies of Pr_t and Sc_t in jets with Pr and Sc substantially different from unity have been carried out by Keagy and Weller [14], Forstall and Shapiro [15], Forstall and Gaylord [16], Sakipov and Temirbaev [17], and Way and Libby [18]. The distribution of small particles in jets displays somewhat similar features; work on flows of this kind has been reviewed by Goldschmidt *et al.* [19] and Lilly [20]. Many of the investigators whose work is reported in [14] to [18] have given only the mean values of the turbulent or effective Prandtl number that were obtained by fitting semi-empirical models to measured profiles of velocity, temperature and concentration.

A number of investigations have been made into pure gaseous jets in air [14, 18], but these include another source of variability, namely, the large density changes across and along the flow. After prolonged mixing with their environment, these flows ultimately merge with the class of flows with dilute concentration, but in practice measurements have been made rather close to the efflux, where the density variation is still large. These remarks apply also to studies of strongly heated air jets, some of which are reviewed by Schubauer and Tchen [21] and Beér and Chigier [22].

Finally, we may note the large amount of work related to the propulsive jets of rockets and jet engines and to the wakes of vehicles moving at high speed through the atmosphere. Typically, these experiments involve high Mach numbers [23, 24], large concentration variations [25–27], coaxial jets [25, 27, 28], or more general outer flows [26]. A variety of attempts to predict these complex flows and the simple ones we shall consider are presented in [2, 29].

2. SURVEY OF DATA

We shall consider variations of Pr_t and Sc_t in wakes and jets whose motion is essentially independent of the variations in temperature or concentration. In the Appendix to this paper it is shown that the turbulent Prandtl and Schmidt numbers can be calculated as

$$Pr_t = \varepsilon_m/\varepsilon_h = \frac{\Delta U}{\Delta T}\frac{\partial T/\partial y}{\partial U/\partial y}, \tag{1}$$

$$Sc_t = \varepsilon_m/\varepsilon_D = \frac{\Delta U}{\Delta C}\frac{\partial C/\partial y}{\partial U/\partial y} \tag{2}$$

for fully turbulent wakes and jets, that is, when the contributions of molecular diffusion are negligible. It is assumed in the derivation that:

(a) the motion is governed by boundary-layer or thin-flow approximations to the equations governing momentum, heat- and mass-transfers,

(b) the profiles of the averaged properties (U, T, C, $\overline{uv}$, $\overline{v\theta}$ and $\overline{vc}$) are of self-preserving form, and

(c) the boundary conditions for the three transfer processes are of similar form.

The existence of a self-preserving pattern of development requires that the flow be independent of external influences such as an inconsistent pattern of external convection or background turbulence. Uniform convection was present in the jet flows studied by Pabst [9], but when the decay of the velocity excess is well advanced this flow should degenerate into a "negative wake", self-preserving as is the normal wake with negative momentum. Background turbulence was found to have a significant effect on the results of Lilly [20], and its role has been investigated analytically by Fink [30].

In the plane wake and round jet the molecular and turbulent transfer processes develop in the same way along the flow. Hence the formulae (1, 2) apply even when molecular transport is significant. The effective values

$$Pr_e = \frac{\nu+\varepsilon_m}{\kappa+\varepsilon_h} \quad \text{and} \quad Sc_e = \frac{\nu+\varepsilon_m}{D+\varepsilon_D} \tag{3}$$

are then determined by these equations. This extension is not possible for the round wake and plane jet; they must be fully turbulent if the results (1, 2) are to be applicable. The turbulent mixing layer cannot be treated using these simple formulae—save in the limiting case in which the bounding velocities are very nearly equal—since a single explicit integration of the governing equations is not possible for this flow.

Figures 1–6 show the variations in turbulent or effective Prandtl and Schmidt numbers for five free-turbulent flow species: round and plane jets and wakes, and the convected round jet. Corresponding profiles of mean velocity and intermittency factor are also

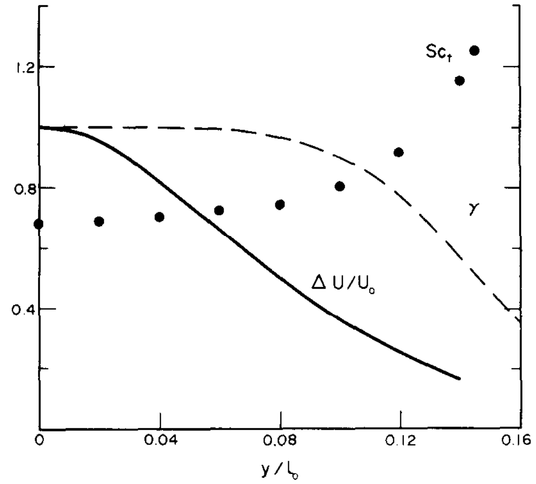

FIG. 1. Dependence of turbulent Schmidt number (solid points) on position within a round jet of air, from results of Hinze and van der Hegge Zijnen [10]. Their velocity variation (solid line) is also shown, together with the intermittency profile (dashed line) of Corrsin and Kistler [31].

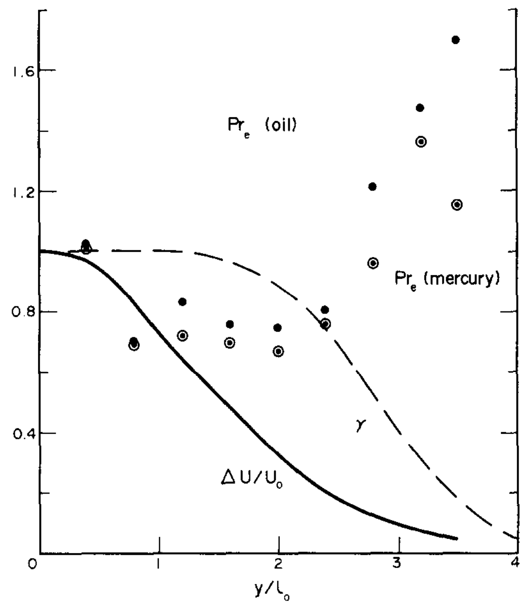

FIG. 2. Effective Prandtl numbers within round jets of oil (solid points) and of mercury (open points), from results of Sakipov and Temirbaev [17]. Their velocity distribution (solid line) is also shown, with the intermittency profile (dashed line) of Corrsin and Kistler [31].

shown where they are available. The diffusivity ratios of Figs. 2 and 4 were calculated by Hinze and van der Hegge Zijnen using the formulae (1–3); the other values have been found by applying these results to published mean profiles. Since many of the measurements were made at sections at which the flow is still moving towards a self-preserving form, the use of this method of analysis is not entirely justified. However, it may be many years before the measurements required for more convincing calculations are carried out. The velocity measurements gven in the figures are those on which

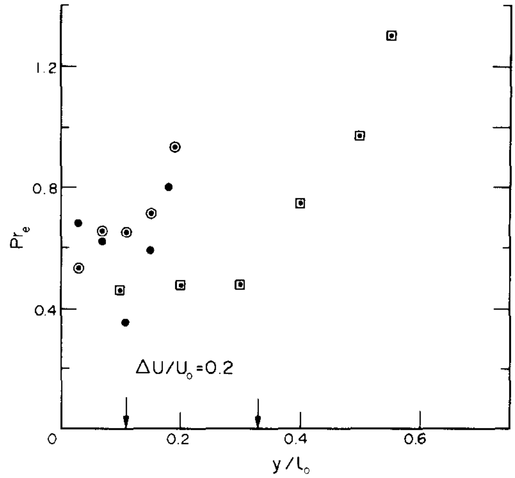

FIG. 3. Effective Prandtl numbers within round convected jets of heated air, from measurements of Pabst [9]: $(U_i - U_1)/U_i = 0.53$ (open squares), 0.74 (solid points) and 0.95 (open points). (U_i is the initial jet velocity, and U_1 is the uniform convection velocity in the outer flow.) The arrows mark the positions where $\Delta U/U_0 = 0.2$; the right-hand arrow relates to $(U_i - U_1)/U_i = 0.53$, and the left-hand arrow to the other two jets.

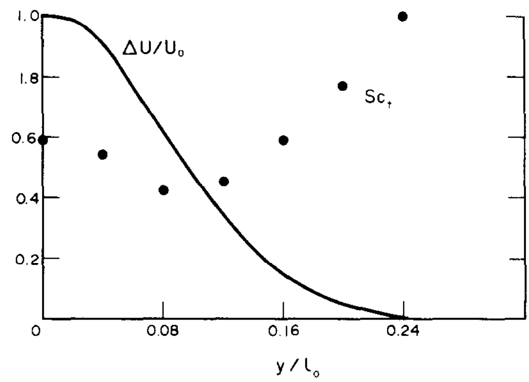

FIG. 4. Turbulent Schmidt numbers (solid points) within a plane jet of air, from measurements of van der Hegge Zijnen [7]. His velocity profile is also shown.

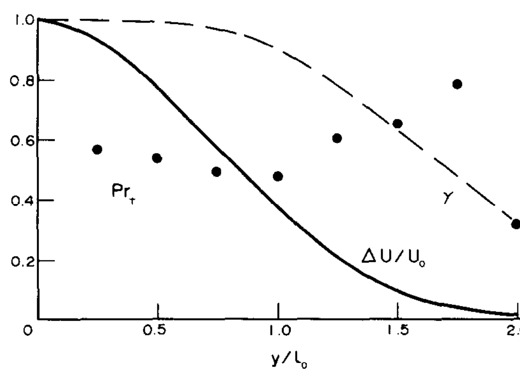

FIG. 5. Turbulent Prandtl numbers (solid points) within a plane wake, from measurements of Reichardt [4]. His velocity distribution (solid line) and the intermittency distribution (dashed line) of Townsend [32] are also shown.

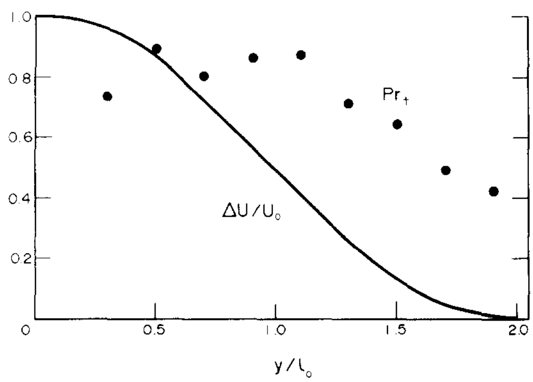

FIG. 6. Turbulent Prandtl numbers (solid points) within a round wake, from measurements of Reichardt and Emshaus [6]. Their velocity distribution is also shown.

the diffusivity calculations were based, but the intermittencies were obtained from separate investigations by Corrsin and Kistler [31] and by Townsend [32].

It is usual to assume that the relationships

$$Pr_t = f(Pr, Re) \quad \text{and} \quad Sc_t = f(Sc, Re) \qquad (4)$$

have exactly the same functional form, and it then follows that the effective values (3) are related in a like manner. As noted earlier, this analogy between heat- and mass-transfers does not extend to large temperature and concentration differentials, nor to the transport of finite particles or long-chain molecules that deviate from or modify the local fluid motion. Nor does it apply exactly in turbulent gas flows, for reasons that have been set out elsewhere (Reynolds [33, 34]). However, in the present context any lack of precision in this analogy will be neglected, in keeping with the neglect of the influence of Reynolds number on Pr_t and Sc_t.

For all of the results presented the basic fluid was air, and Pr, $Sc = 0.3$–1.0, save for the measurements of Sakipov and Temirbaev (Fig. 2), which relate to jets in oil ($Pr \sim 100$) and mercury ($Pr = 0.024$). Hence for the majority of the cases considered the analogy between heat and mass transfer will be nearly perfect, so far as the dependence on Pr and Sc is concerned, provided that equations (4) are correct. Figures 1 and 2 relate to jets in fluids with very different values of molecular Pr, yet the results are very similar over much of the shear layer. The differences in the outer part of the flow can readily be explained by reference to the increasing role of the molecular diffusivities there. An attempt will be made in Section 4 to rationalize the similarity of the inner portions of the profiles.

The experiments of Pabst (Fig. 3) differ from the others in two ways, first in having a uniform outer convection velocity which provides turbulence beyond the shear layer and slows the development towards a self-preserving form and, secondly, in having a rather large ratio of jet temperature to ambient temperature (the ratio of absolute temperatures was about two for these tests). These features may explain the exceptionally low values of Pr_e achieved in the body of the shear layer; other experiments with relatively light jets [14, 18, 27] display the same feature.

Some of the data given in Figs. 1–6 indicate a small increase in the diffusivity ratio very near the axis, while others suggest a nearly uniform value in the nearly-always-turbulent region. In view of the difficulty of determining the required slopes accurately near the axis, the two kinds of behaviour are equally plausible. For most engineering calculations it should be sufficient to adopt a uniform value of Pr_t or Sc_t in this central region, but this is probably no more than a convenient approximation.

3. THE ROLE OF INTERMITTENCY

The most striking feature of the results under consideration is the large difference between the diffusivity ratio in the body of the shear layer and those near the outer boundary. Typically, the region in which the diffusivity ratio is substantially different from the body value is that for which the intermittency factor $\gamma < 0.7$ and the velocity defect satisfies $\Delta U/U_0 < 0.2$. If the outer limit of the shear layer is arbitrarily defined by $\gamma = 0.2$, this edge region is found to occupy nearly one-half the cross-sectional area of the layer.

In only one case, that of the round wake (Fig. 6), are the values in the edge region consistently less than those in the body of the flow. The turbulent Prandtl numbers obtained from the round-wake results of Hall and Hislop [5] are identical, within the scatter arising from the numerical differentiation, to those plotted. In this connexion we may note that the measurements of Fage and Falkner [3] for the plane wake behind a lenticular cylinder are closely represented by $Pr_t = 0.5$, with perhaps a small drop at the very edge of the region in which measurements were made. This pattern is intermediate to the wake results of Figs. 5 and 6.

For the jet flows the relationship between body and edge values is clearer, with the turbulent moduli rising rapidly in the edge region in every case.

Table 1 gives the diffusivity ratios typical of the body and edge region of each kind of shear layer: B is the mean value for the central part of the flow; E is the value for $\gamma = \frac{1}{2}$. The tabled values are thought to be correct to ± 0.05, with B for the round jet more precisely known, since this flow has been studied more often.

Table 1. Values of Pr_t, Sc_t for Pr, Sc close to unity

Flow species	B, body value ($\gamma = 1$)	Pr_t or Sc_t E, edge value ($\gamma = \frac{1}{2}$)	N, outer value ($\gamma \to 0$)
Round jet	0.73	1.2	1.7
Round wake	0.8	0.55	0.3
Convected round jet	0.6	1.1	1.6
Plane jet	0.5	0.9	1.3
Plane wake	0.5	0.6	0.7

We note that the body values for the round or axisymmetric flows are larger than those for plane or two-dimensional flows. On surveying data relating to the central region or turbulent core of flows within closed channels, Groenhoff [35] reached a similar conclusion: Pr_t, $Sc_t \simeq 0.8$ for flow in a round pipe, and Pr_t, $Sc_t \simeq 0.5$ for flow in a plane channel, in each case for Pr, Sc near unity. On the other hand, Table 1 shows that the edge value for free turbulence depends on whether the flow is a jet or a wake, with jets having values about twice those for wakes. Blom [36], in surveying measurements of Pr_t for turbulent boundary layers, found behaviour more like that observed here for jets, with higher values in the outer turbulent region of the layer.

Rebollo's [13] study of the turbulent mixing layer between pure nitrogen and pure helium gave Pr_t variations generally similar to those considered here, with values 0.2–0.4 in the always turbulent region, rising to 1.2–1.4 in highly intermittent conditions. The very low body values may be ascribed in part to the two-dimensional nature of the basic flow (compare with Table 1), and in part to the large density variation, which commonly gives rise to low central values of Pr_t, as was noted earlier.

From the practical point of view cross-flow variations in Pr_t and Sc_t are important because the assumption of a uniform, low value will lead to an overestimate of the width of the temperature or concentration profile relative to the velocity distribution, that is, to the prediction of excessive values of ΔT and ΔC in the outer part of the flow. From the fundamental point of view the variations are important in suggesting that a quite different treatment will be required for the analysis of intermittently turbulent flows. Some steps in this direction have recently been taken by Libby [37], but calculations of heat- and mass-transfer have not yet taken explicit account of intermittency. In a recent survey of methods of predicting turbulent Prandtl and Schmidt numbers [34], only a few were found that had any pretensions to dealing with free-turbulent flows (Tyldesley [38] and Buleev [39]). Not surprisingly, more attention has been given to wall flows, which occur in a wider range of applications of technical importance.

A satisfactory treatment of transfers of heat and matter within free-turbulent flows will undoubtedly utilize transport equations rather than phenomenological entities such as eddy diffusivities and Prandtl and Schmidt numbers incorporating them. Nevertheless, it may be convenient for engineering calculations to have available some simple rules connecting the distributions of mean velocity and other mean properties. For those cases in which the distributions of both Pr_t and γ are known, the edge region in which Pr_t departs from the body value corresponds roughly to the region in which γ departs from unity. A simplistic explanation is provided by the hypothesis that the actively turbulent fluid is characterized by one value of the turbulent Prandtl number (B) and the non-vortical outer fluid by another (N, say). This postulate is expressed in the formula

$$Pr_t = \gamma B + (1-\gamma) N \tag{5}$$

but this may prove useful as an interpolation formula even if the idea behind it proves to be incorrect.

It is reasonable to expect that $B = f(Pr, Re)$, while N may be primarily a function of the statistical kinematics of the interface and hence virtually independent of Pr and Re. (The nature of the functional dependence of B will be considered in Section 4 below.) Hence the superposition (5) provides a means of estimating the variation of the Pr_t-profile with changes in the molecular value Pr.

The outer value N can be estimated in terms of numbers already given in Table 1 as

$$N = 2E - B \tag{6}$$

since E is the value of Pr_t pertaining when $\gamma = \frac{1}{2}$. The results obtained from this equation are given in the final column of Table 1 for the five flows for which experimental data are available. They suggest the ultimate levels that may be attained by Pr_t, Sc_t at the edge of a shear layer, where calculations using equations (1, 2) are at best inaccurate and in practice impossible since data are not available.

The values of N for the three jet flows are not very different, being grouped around $Pr_t = 1.5$. The values for the two wake flows differ markedly from one another. Moreover, the variation of Pr_t near the edge of the plane wake is rather uncertain, and may conflict with the simple proposal (5). We may conclude, however, that the values of N for wakes are of the order of one-half to one-third those for jets.

4. GENERAL NATURE OF DEPENDENCE ON MOLECULAR PROPERTIES

One feature of the experimental results for a round jet on which some light can be cast is the apparent near independence of the body value of Pr_e of the molecular value Pr; compare Figs. 1 and 2. From the many formulae (see Reynolds [34]) purporting to indicate the dependence $Pr_t = f(Pr, Re)$, we select the simple result

$$Pr_t = \frac{1 + C_1 Pr^{-\frac{1}{2}} Re^{-\frac{1}{2}}}{1 + C_2 Re^{-\frac{1}{2}}}. \tag{7}$$

In [33] the constants $C_1 = 86$ and $C_2 = 200$ are suggested as representative of the cores of pipe flows, for which $Re = Re_d = U_a d/\nu$, with U_a the average velocity and d the pipe diameter. In order to estimate the trends in free turbulence, we modify these constants to

$$C_1 = 30 \quad \text{and} \quad C_2 = 50 \tag{8}$$

and take

$$Re = Re_0 = U_0 l_0/\nu$$

where U_0 is the velocity change across the layer and l_0 is the length scale measured from the axis to the point at which $\Delta U/U_0 = \frac{1}{2}$, say.

The values (8) can be justified as follows. To account roughly for the alterations in the velocity and flow-width scales, we might substitute $Re_d = 8Re_0$. However, the resulting values, $C_1 = 30$ and $C_2 = 70$, lead to unrealistically low predictions of Pr_t for the much-studied case of a round jet in air. Accordingly, the ratio of the constants has been altered in order that the formula represent this case with reasonable accuracy.

We turn now to the effective value Pr_e. From equations (3) we have

$$Pr_e = Pr_t \frac{1+\nu/\varepsilon_m}{1+(\nu/\varepsilon_m)Pr_t/Pr}. \quad (9)$$

For free turbulence the difference between Pr_e and Pr_t can be estimated, for any combination of Pr and Re_0, by taking the eddy viscosity to be uniform within the turbulent body of the flow. Considerations of self-preservation and dimensional homogeneity lead to

$$U_0\, l_0/\varepsilon_m = R_0$$

with R_0 a constant for any one species of free-turbulent flow. Hence

$$\nu/\varepsilon_m = R_0/Re_0. \quad (10)$$

For a specified value of the flow constant R_0, equations (7–10) now define $Pr_e = f(Pr, Re_0)$. Values of R_0 appropriate to a number of flow species are given in Table 7.2 of [32].

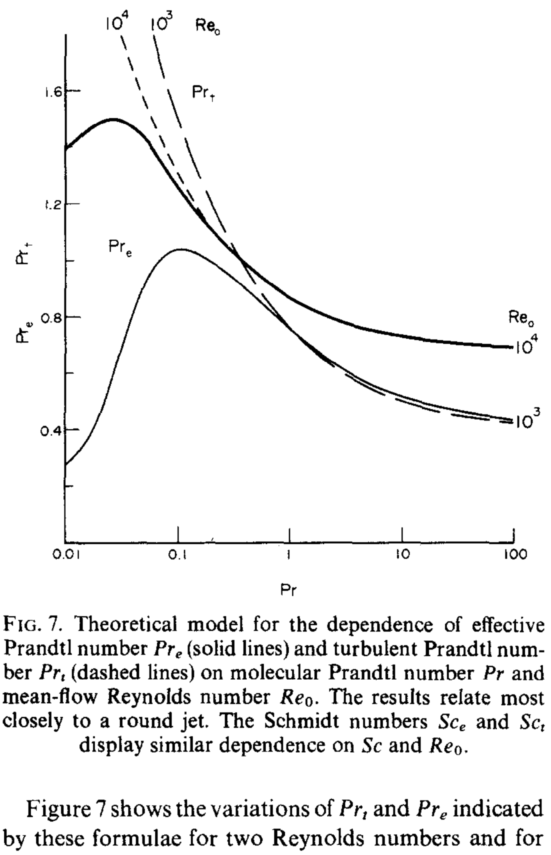

FIG. 7. Theoretical model for the dependence of effective Prandtl number Pr_e (solid lines) and turbulent Prandtl number Pr_t (dashed lines) on molecular Prandtl number Pr and mean-flow Reynolds number Re_0. The results relate most closely to a round jet. The Schmidt numbers Sc_e and Sc_t display similar dependence on Sc and Re_0.

Figure 7 shows the variations of Pr_t and Pr_e indicated by these formulae for two Reynolds numbers and for $R_0 = 35$, the value appropriate to a round jet. These results are probably incorrect in detail, but from them we can draw some conclusions broadly applicable to this and other free-turbulent flows (and applicable also to mass transfer if Pr is replaced by Sc): (a) Pr_e and Pr_t are nearly equal for $Pr > 0.5$. This range encompasses most realistic cases (liquid–metal heat-transfer excepted), but the conclusion must be applied with caution to the outermost, highly intermittent part of the flow. (b) For $Pr > 0.5$, Pr_t increases with increasing Reynolds number; Pr_e always increases with Reynolds number. (c) In an intermediate range, $0.3 < Pr < 3$, for example, the effect of Reynolds number is small. This range includes almost all cases of heat transfer and diffusion in gases, and also some examples of heat transfer in liquids. (d) The band $Pr_e = 0.7 \pm 0.2$ encompasses most realistic situations, since there are few applications near $Pr, Sc = 0.1$.

The final conclusion shows how it is possible for nearly equal body values of Pr_e to be obtained for $Pr = 0.02$, 0.7 and 100, as appears to be the case in Figs. 1 and 2. An examination of equations (7, 9) shows that the limited range of Pr_e, Sc_e is the result of opposing effects of the direct dependence on Pr and that implicit in Pr_t.

5. CONCLUSIONS

(a) Turbulent Prandtl and Schmidt numbers vary by a factor of two or more across most jets and wakes; the exception is the plane wake where the variation is probably small.

(b) For jets the outer values of Pr_t and Sc_t are around 1.5; failure to account for this will lead to the prediction of excessive values of ΔT and ΔC in the outer part of the flow.

(c) Very large changes in the molecular values Pr and Sc have a marked influence on the effective values Pr_e and Sc_e in the outer part of the shear layer. In the body of the layer the changes in Pr_e and Sc_e are less marked, since the direct dependence on Pr or Sc is opposed by the indirect dependence through Pr_t and Sc_t.

(d) These conclusions relate to nearly self-preserving flows at substantially uniform density. The evidence is meagre for free-turbulent flows with large density variations, convected by possibly complex streams, but it indicates that even larger changes in Pr_t and Sc_t occur in such flows.

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APPENDIX: RESULTS FOR SELF-PRESERVING FREE-TURBULENT FLOWS

For simplicity, we consider the particular case of a heated plane wake, for which the momentum and thermal energy equations are

$$U_1\,\partial U/\partial x = -\partial \overline{uv}/\partial y + \nu\,\partial^2 U/\partial y^2$$

and

$$U_1\,\partial H/\partial x = -\partial \overline{vh}/\partial y + \kappa\,\partial^2 H/\partial y^2$$

subject to the usual boundary-layer and small-deficit approximations. Assuming a self-preserving mode of development, we take

$$U = U_1 + U_0 f(\eta), \quad \overline{uv} = U_0^2 g_{12}(\eta),$$
$$H = H_1 + H_0 j(\eta) \quad \text{and} \quad \overline{vh} = U_0 H_0 k_2(\eta)$$

where $\eta = y/l_0$ and l_0, U_0 and H_0 are the scales for width, velocity and enthalpy.

The governing equations now become

$$C_1(f+\eta f') = -g_{12} + f''/Re_0$$

and (A1)

$$C_1(j+\eta j') = -k_2' + j''/Pe_0$$

with

$$Re_0 = U_0 l_0/\nu \quad \text{and} \quad Pe_0 = U_0 = U_0 l_0/\kappa \qquad \text{(A2)}$$

and

$$C_1 = -(U_1/U_0)\,\mathrm{d}l_0/\mathrm{d}x$$

all constants in this situation, since the momentum-flux deficit is uniform along the flow.

Subject to the conditions of symmetry

$$f' = g_{12} = j' = k_2 = 0 \quad \text{at} \quad \eta = 0$$

integration of the two equations gives

$$C_1 \eta f = -g_{12} + f'/Re_0$$

and (A3)

$$C_1 \eta j = -k_2 + j'/Pe_0$$

whence

$$\frac{f}{j} = \frac{-g_{12} + f'/Re_0}{-k_2 + j'/Pe_0}.$$

Introducing the eddy diffusivities, we have

$$\frac{f}{j} = \frac{\varepsilon_m + \nu}{\varepsilon_h + \kappa}\frac{f'}{j'} = Pr_e \frac{f'}{j'}$$

or, in terms of dimensioned quantities

$$Pr_e = \frac{U - U_1}{T - T_1}\frac{\partial T/\partial y}{\partial U/\partial y}. \qquad \text{(A4)}$$

The derivation of this result requires that the scaled equations (A1) be ordinary differential equations, that they be integrable to give direct relationships (A3) between fluxes and mean properties, and that the scaled convective derivatives have the same structure. For the particular flow considered above, Re_0 and Pe_0 are constants, since the momentum deficit is invariant along the flow. This is also true for the round jet, but not for the plane jet, round wake and plane mixing layer. However, the molecular terms can be dropped without significant error, provided that $Pr \not\ll 1$, and that Re_0 is large enough. The resulting equations for the plane jet and round wake can then be integrated, and the result (A4) is recovered, but with Pr_t replacing Pr_e. The modified equation (A4) is then equivalent to equation (1).

The momentum and thermal-energy invariants for the five self-preserving flows (round wake and jet, and plane wake, jet and mixing layer) ensure that the ratio of scales U_0/T_0 (here T_0 is a time scale) is constant along each flow. Hence the scaled convective derivatives for velocity, enthalpy and concentration have the same form in the three equations governing an individual flow, as is the case in equations (A1).

We conclude that equations (1, 2) apply to round wakes and plane jets, unless Pr or $Sc \ll 1$ or $\gamma \to 0$, and that the modification indicated in equation (A4) holds for the plane wake and round jet, even in these extreme cases. Equations (1, 2) also apply to self-preserving patterns of development in suitably tailored streaming flows, but not for an arbitrary outer flow. In particular, they do not apply for a uniform outer stream, except in the wake limit, $U_0/U_1 \ll 1$.

For the self-preserving mixing layer, the governing equations, with molecular contributions rejected, reduce to

$$FF'' = g'_{12} \quad \text{and} \quad Fj' = k'_2$$

where F is a scaled stream function, such that $U = U_0 F'$. Although these equations have a simple structure, they cannot be integrated explicitly in the manner of equations (A2), and no formula can be found giving Pr_t directly in terms of the variations of mean velocity and temperature.

VARIATION DES NOMBRES DE PRANDTL ET DE SCHMIDT TURBULENTS DANS LES SILLAGES ET DANS LES JETS

Résumé—On s'interesse aux valeurs d'origine expérimentale des nombres de Prandtl et de Schmidt turbulents et effectifs mesurées dans les écoulements libres turbulents. Une attention particulière est donnée aux variations de ces grandeurs à travers des écoulements de densité presque uniforme. Dans les régions fortement intermittentes les valeurs obtenues peuvent différer, par un facteur deux, des valeurs relatives au coeur de la couche cisaillée. Le rôle de l'intermittence et l'influence des diffusivités moléculaires et du nombre de Reynolds sont étudiés à l'aide de modèles analytiques simples.

DIE VERÄNDERUNG DER TURBULENTEN PRANDTL- UND SCHMIDT-ZAHL IN NACHLAUF- UND DÜSENSTRÖMUNGEN

Zusammenfassung—Die Betrachtung erstreckt sich auf experimentell ermittelte Werte der turbulenten und effektiven Prandtl- und Schmidt-Zahl in freien, turbulenten Strömungen. Besonders betrachtet werden die Veränderungen quer zu Strömungen von nahezu einheitlicher Dichte. In stark intermittierenden Bereichen können sich diese Moduln um einen Faktor 2 ändern gegenüber den Werten in einer Scherschicht. Nach einfachen analytischen Modellen wird der Einfluß dieser Schwankungen untersucht und der Einfluß von molekularer Diffusivität und der Reynolds-Zahl festgestellt.

ИЗМЕНЕНИЕ ТУРБУЛЕНТНЫХ ЧИСЕЛ ПРАНДТЛЯ И ШМИДТА В СЛЕДАХ И СТРУЯХ

Аннотация — Рассматриваются экспериментально найденные турбулентные и эффективные числа Прандтля и Шмидта для свободноконвективных потоков. Особое внимание уделено изменениям этих параметров в потоках с почти однородной плотностью. В областях с высокой степенью перемежаемости параметры могут отличаться в два раза от их значений в пограничном слое около тела. Используются простые аналитические модели для исследования вклада перемежаемости и зависимости вышеуказанных параметров от молекулярных коэффициентов диффузии и числа Рейнольдса.